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Lie Algebras and Representations of Continuous Groups

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Abstract

It is shown that every finitely generated continuous group has a subgroup generated by its infinitesimal transformations. This subgroup has a group algebra which is the Lie algebra of the group. By obtaining complete systems in the Lie algebra and complete rectangular arrays, it is shown that these can yield matrix representations of the continuous group. Illustrative examples are given for the rotation groups and for the full linear groups. It would seem that all the finite motion representations can be obtained by these methods, including spin representations of rotation groups. But the completeness of the method is not here demonstrated.

1. *Introduction*

The basic method of finding the matrix representations of finite groups is to form the group algebra over the complex numbers and to analyze this into a direct sum of total matrix algebras. These total matrix algebras give a complete set of irreducible matrix representations.

Extending the theory to continuous groups it has been thought [Littlewood (1958), p. 283] that this method could not be applied since the number of group elements in a continuous group is more than countable, and an algebra with a transcountable basis is hardly analyzable.

However, this objection does not take into account the property of continuity, and the restrictions which this implies. In effect, these restrictions can reduce the basis of the algebra from a transcountable to a countable set.

It is found that for any finitely generated continuous group a group algebra can be formed with a countable basis, and that this is, in fact, the Lie algebra of the group.

Methods will be obtained for finding matrix representations of the Lie algebra. These would seem to yield all the matrix representations of the continuous group, although no proof is given here of the completeness.

2. *Cyclic Subgroup of a Continuous Group*

Any continuous group contains infinitesimal elements. Any such infinitesimal element generates a cyclic group which consist of all real powers of an element S. If the group is compact, some finite powers of S will be equal to the identity. Replacement of S by some power of S will ensure that

$$S^{2\pi} = I$$

and that 2π is the least nonzero index that satisfies this equation.

The theory, however, is not restricted to compact groups. The assumption of continuity is contained in the hypothesis that in the group algebra

$$S^{1/n} - I$$

tends to zero as $n \to \infty$. Putting

$$S^{1/n} = I + x/n + O(1/n^2)$$

and making $n \to \infty$ we obtain

$$S = \exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$S^a = \exp (ax) = 1 + ax + \frac{a^2x^2}{2!} + \cdots$$

We thus obtain an algebra, the algebra of power series in x, such that for every element S^a of the cyclic group there corresponds an element of the algebra.

Further, given any set of n group elements $S^{a_1}, S^{a_2}, \ldots, S^{a_n}$, the corresponding elements of the group algebra are linearly independent, for the determinant of coefficients of the lowest n powers of x is

$$\left| \frac{a_s^{t-1}}{(t-1)!} \right| = \frac{1}{1!2! \cdots (n-1)!} \prod_{i<j} (a_i - a_j)$$

which is nonzero.

Hence the algebra of power series in x is a group algebra for the cyclic group. If the group is compact it would be necessary to take the system of residues to modulus $[\exp (2\pi x) - 1]$.

The matrix representations of an Abelian group are one dimensional. The representations of the compact group are of the form

$$S \to e^{ir}$$

for integral r, positive, zero, or negative, whence

$$x \to ir$$

Many-valued representations can be obtained by making r a rational number. For noncompact groups, x can correspond to any number real or complex.

3. *Continuous Groups*

Let H denote a finitely generated continuous group. The infinitesimal elements generate a subgroup G possibly identical with H. The group G will be called a *Lie group*.

Let G have m linearly independent infinitesimal elements $S_i = I + x_i/n$, $(n \to \infty; i = 1, 2, \ldots, m)$. The group G is generated by the cyclic subgroups corresponding to the various x_i's. Hence the group algebra of G is generated by the group algebras of the cyclic subgroups and consists of power series in all the x_i's. The term x_i, x_j do not necessarily commute, and the deviation from commutativity will be examined.

The commutator $S_j^{-1}S_iS_jS_i^{-1}$ belongs to the group, and hence to the group algebra. Then

$$S_j^{-1}S_iS_jS_i^{-1} = S_j^{-1}(S_iS_j - S_jS_i)S_i^{-1} + I$$

But

$$\begin{aligned} S_iS_j - S_jS_i &= \left(I + \frac{x_i}{n}\right)\left(I + \frac{x_j}{n}\right) - \left(I + \frac{x_j}{n}\right)\left(I + \frac{x_i}{n}\right) \\ &= (1/n^2)(x_ix_j - x_jx_i) \end{aligned}$$

Hence

$$S_j^{-1}S_iS_jS_i^{-1} = I + (1/n^2)(x_ix_j - x_jx_i) + O(1/n^3)$$

It follows that $I + (1/n^2)(x_ix_j - x_jx_i)$ is an infinitesimal group element and is thus expressible linearly in terms of $x_1, x_2, \ldots, x_m$, in the form

$$x_ix_j - x_jx_i = \sum \gamma_{ijk}x_k \tag{3.1}$$

> *Theorem I.* Every finitely generated continuous group has a subgroup (a Lie group) generated by the infinitesimal elements. There is a group algebra of this subgroup which consists of power series in m variables $x_1, x_2, \ldots, x_m$ subject to relations of the form
>
> $$x_ix_j - x_jx_i = \sum \gamma_{ijk}x_k$$

This group algebra is identical with the Lie algebra of the group as commonly defined. Expressions of the form $xy - yx$ are in such common use that a special terminology is commonly used.

Poisson Bracket Definition. The Poisson Bracket of two quantities x, y is defined as

$$[x, y] = xy - yx$$

Clearly

$$[y, x] = -[x, y]$$

4. *The Lie Algebra*

Complete Systems. Let $f_1, f_2, \ldots, f_r$ be a set of r elements of the Lie algebra, i.e., a set of r polynomials in the basal elements $x_1, \ldots, x_m$. Then, if each Poisson bracket

$$[x_i, f_j]$$

is expressible linearly in terms of the set $f_1, \ldots, f_r$, then this set is said to form a *complete system.*

Suppose that

$$[x_i, f_j] = -\sum \Gamma_{ijk} f_k$$

Express the elements of the complete system as a column vector

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{bmatrix} = [f_s]$$

The letters s, t will be reserved to indicate the row and column, respectively, of a typical element of a matrix or vector. Then

$$\begin{aligned} [x_i, F] = x_i F - F x_i = -[\Gamma_{isr} f_r] &= -[\Gamma_{ist}][f_s] \\ &= -X_i F \end{aligned}$$

where $X_i = [\Gamma_{ist}]$, a square matrix with scalar elements. Then

$$[x_j, [x_i, F]] = [x_j, -X_i F] = -X_i [x_j, F] = X_i X_j F$$
$$x_j x_i F - x_j F x_i - x_i F x_j + F x_i x_j = X_i X_j F$$

Interchanging i and j and subtracting we obtain

$$(x_j x_i - x_i x_j)F - F(x_j x_i - x_i x_j) = (X_i X_j - X_j X_i)F$$

assuming that

$$x_i x_j - x_j x_i = \sum \gamma_{ijk} x_k$$
$$\begin{aligned} -\sum \gamma_{ijk}(x_k F - F x_k) &= (X_i X_j - X_j X_i)F \\ &= \sum \gamma_{ijk} X_k F \end{aligned}$$

It follows that

$$[X_i, X_j] = \sum \gamma_{ijk} X_k$$

Theorem II. Any complete system $f_1, f_2, \ldots, f_r$ in a Lie algebra such that

$$[x_i, f_j] = -\sum \Gamma_{ijk} f_k$$

yields a matrix representation of the Lie algebra and of the Lie group for which

$$x_i \to [\Gamma_{ist}]$$

Construction of Complete Systems, Symmetrized Products. Consider the monomial expression of degree r

$$x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} \quad (a_1 + a_2 + \cdots + a_m = r)$$

There are $r!/(a_1!a_2!\cdots)$ distinct monomial expressions which can be obtained from this by permuting the factors. The symmetrized product is defined as the arithmetic mean of these and is denoted by

$$(x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}) = \frac{a_1!\,a_2!\cdots a_m!}{r!} \sum x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$$

Since $x_i x_j - x_j x_i$ is expressible linearly in terms of $x_1, \ldots, x_m$, the difference between any two terms of a symmetrized product is expressible in terms of polynomials of lower degree, e.g.,

$$x_i x_j x_k - x_j x_k x_i = (x_i x_j - x_j x_i) x_k + x_j (x_i x_k - x_k x_i)$$

which is expressible in terms of quadratics. It follows that any polynomial of degree r is expressible in terms of the symmetrized products of degree r, and in the polynomials of lower degree.

> *Lemma II.* If f_i is a symmetrized product of degree r, then $[x_j, f_i]$ is expressible linearly in terms of symmetrized products of degree r.

To prove this, note that if any monomial expression P of degree r is expressible in r ways as

$$P = P' x_k P''$$

then

$$[x_j, P] = \sum P' [x_j, x_k] P''$$

A symmetrizing operation on P will simultaneously symmetrize $P'[x_j, x_k]P''$, and the proof of the lemma will follow.

As a consequence of these two lemmas we have

> *Theorem III.* The symmetrized products of degree r form a complete system, and yield a matrix representation of the Lie algebra and of the Lie group.

This representation may be reducible.

5. *The Three-Variable Rotation Group*

As an example of the method consider the three-variable rotation group. Using, u, v, w to denote the basal elements of the Lie algebra, the equations are

$$uv - vu = w, \qquad vw - wv = u, \qquad wu - uw = v$$

Then

$$u\begin{bmatrix} u \\ v \\ w \end{bmatrix} - \begin{bmatrix} u \\ v \\ w \end{bmatrix} u = \begin{bmatrix} 0 \\ w \\ -v \end{bmatrix} = \begin{bmatrix} 0 & & \\ & & 1 \\ & -1 & \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Hence

$$u \to \begin{bmatrix} 0 & & \\ & & -1 \\ & 1 & \end{bmatrix}$$

and similarly

$$v \to \begin{bmatrix} & & 1 \\ & 0 & \\ -1 & & \end{bmatrix} \qquad w \to \begin{bmatrix} & -1 & \\ 1 & & \\ & & 0 \end{bmatrix}$$

The matrix corresponding to a rotation in the x_2x_3 plane through an angle θ is

$$\exp(\theta a) = \begin{bmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{bmatrix}$$

Concerning the symmetrized products of degree 2, clearly

$$u^2 + v^2 + w^2$$

is an invariant, e.g.,

$$u(u^2 + v^2 + w^2) - (u^2 + v^2 + w^2)u = 0 + wv + vw - vw - vw = 0$$

Arranging the other five terms in a vector

$$\phi = \begin{bmatrix} \frac{1}{\sqrt{3}}(-u^2 - v^2 + 2w^2 \\ u^2 - v^2 \\ vw + wv \\ wu + uw \\ uv + vu \end{bmatrix}$$

the calculation of $[u, \phi]$, $[v, \phi]$, and $[w, \phi]$ gives

$$u \rightarrow \begin{bmatrix} . & . & \sqrt{3} & . & . \\ . & . & 1 & . & . \\ -\sqrt{3} & -1 & . & . & . \\ . & . & . & . & 1 \\ . & . & . & -1 & . \end{bmatrix}$$

$$v \rightarrow \begin{bmatrix} . & . & . & -\sqrt{3} & . \\ . & . & . & 1 & . \\ . & . & . & . & -1 \\ \sqrt{3} & -1 & . & . & . \\ . & . & 1 & . & . \end{bmatrix}$$

$$w \rightarrow \begin{bmatrix} 0 & . & . & . & . \\ . & . & . & . & -2 \\ . & . & . & 1 & . \\ . & . & -1 & . & . \\ . & 2 & . & . & . \end{bmatrix}$$

$$u^2 + v^2 + w^2 \rightarrow -6I$$

$$\exp(u\theta) \rightarrow \begin{bmatrix} (1 + 3\cos 2\theta)/4, & \sqrt{3}(-1 + \cos 2\theta)/4, & \sqrt{3}\sin 2\theta/2, & & \\ \sqrt{3}(-1 + \cos 2\theta)/4, & (3 + \cos 2\theta)4, & \sin 2\theta/2, & & \\ -\sqrt{3}\sin 2\theta/2, & -\sin 2\theta/2, & \cos 2\theta, & & \\ & & & \cos\theta, & \sin\theta \\ & & & -\sin\theta, & \cos\theta \end{bmatrix}$$

The other true representations of the three-variable rotation group can be obtained in a similar manner.

The method gives all true representations of the three-variable rotation group, but it does not give the spin representations.

In four variables, using $P_{ij}(i, j = 1, 2, 3, 4)$ as the basal elements of the Lie algebra, the linear elements can be divided into two sets:

$$P_{12} + P_{34}, \qquad P_{13} + P_{42}, \qquad P_{14} + P_{23}$$

and

$$P_{12} - P_{34}, \qquad P_{13} - P_{42}, \qquad P_{14} - P_{23}$$

The two sets commute with each other, and each set is isomorphic with the three-variable Lie algebra.

In five variables the linear elements P_{ij} lead to the representation of type $[1^2]$. The terms of degree 2 are reducible, and putting

$$q_1 = P_{23}P_{45} + P_{24}P_{53} + P_{25}P_{34}$$

with similar definitions of q_2, q_3, q_4, q_5, these form a complete set and yield the representation of type $[1]$. The generalization to give the representation of type $[\lambda_1, \lambda_2]$ is not obvious, but no doubt could be obtained.

It seems likely that all true representations of rotation groups could be obtained in this way, although obtaining the general case would not appear to be easy. The spin representations cannot be obtained.

The deficiency of the method becomes obvious when it is applied to the full linear group. For example, in three variables it gives only the self-dual representations.

Clearly some generalization of the method is desirable. Such a generalization will now be obtained by arranging the elements of a complete set in a square, or more generally in a rectangular array, instead of as a column vector. This new method would seem to yield all matrix representations, although proof of completeness in the general case has not yet been obtained.

6. *Complete Rectangular Arrays*

Suppose that a complete set of Lie algebra elements can be arranged in a rectangular array

$$\phi = \begin{bmatrix} u_{11}, & u_{12}, & \ldots, & u_{1n} \\ u_{21}, & \ldots, & & \\ \ldots & & & \\ u_{m1}, & \ldots, & & u_{mn} \end{bmatrix}$$

in such a way that for each basal Lie algebra element x_i there exist scalar matrices X_i and Y_i such that

$$x_i\phi - \phi x_i = X_i\phi - \phi Y_i$$

Then ϕ is said to be a *complete rectangular array.*

Theorem IV. If A and B are nonsingular scalar matrices, and ϕ is a complete rectangular array, then so also is

$$\phi' = A\phi B$$

If

$$x_i\phi - \phi x_i = X_i\phi - \phi Y_i$$

then

$$\begin{aligned} x_i\phi' - \phi' x_i &= A(X_i\phi - \phi Y_i)B \\ &= (AX_iA^{-1})\phi' - \phi'(B^{-1}Y_iB) \\ &= X_i'\phi' - \phi' Y_i' \end{aligned}$$

The effect of replacing ϕ by $A\phi B$ is to replace X_i and Y_i by transforms, and thus X_i and Y_i may be transformed independently in any desired manner.

Reducibility. The array ϕ is said to be *irreducible* if there do not exist scalar matrices A, B, apart from the null matrix and scalar multiples of the identity matrix, such that

$$A\phi = \phi B$$

If, on the contrary, the condition fails and such matrices A, B do exist, then

$$A^2\phi = A\phi B = \phi B^2$$

and if $f(\lambda)$ represents any polynomial,

$$f(A)\phi = \phi f(B)$$

Thus $f(B) = 0$ if and only if $f(A) = 0$. The two matrices A, B must possess the same reduced characteristic equation.

Suppose that this reduced characteristic equation $f(\lambda) = 0$ possesses at least two distinct roots. Then $f(\lambda)$ can be factorized into two coprime factors:

$$f(\lambda) \equiv f_1(\lambda)f_2(\lambda)$$

From Euclid's algorithms

$$F_1(\lambda)f_1(\lambda) + F_2(\lambda)f_2(\lambda) = 1$$

there exist polynomials $l_1(\lambda) = F_1(\lambda)f_1(\lambda)$, $l_2(\lambda) = F_2(\lambda)f_2(\lambda)$ such that $l_1(\lambda)$ and $l_2(\lambda)$ are idempotent, and

$$\left.\begin{aligned} 1 &\equiv l_1(\lambda) + l_2(\lambda) \\ 0 &= l_1(\lambda)l_2(\lambda) \end{aligned}\right\} \bmod \lambda f(\lambda)$$

Hence $l_1(A), l_2(A), l_1(B), l_2(B)$ are idempotent matrices such that

$$\phi = [l_1(A) + l_2(A)]\,\phi\,[l_1(B) + l_2(B)] = l_1(A)\phi l_1(B) + l_2(A)\phi l_2(B)$$

Since any idempotent matrix can be transformed into the form diag $[1^r 0^{n-r}]$, both A and B can be so transformed and ϕ is thus transformed into the form

$$\phi = \left[\begin{array}{c|c} \phi_1 & 0 \\ \hline 0 & \phi_2 \end{array}\right]$$

Thus ϕ is reducible in the ordinary sense.

The only other possibility is that $f(\lambda)$ has but a single root, i.e., is of the form $(\lambda - \lambda_0)^r = 0$. Replacing $A - \lambda_0 I$ by A, it can be arranged so that A, and hence also B, is nilpotent.

There is a power of A, A' which is not zero, but is such that

$$A'^2 = 0$$

$$A'\phi = \phi B'$$

$$B'^2 = 0$$

Then A' and similarly B' can be transformed into the form

$$\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where 0 and I represent blocked matrices.

If the elements of ϕ are similarly blocked as

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}$$

then $A'\phi = \phi B'$ shows that

$$\phi_{11} = \phi_{22},\ \phi_{21} = \phi_{31} = \phi_{23} = 0$$

and

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ 0 & \phi_{11} & 0 \\ 0 & \phi_{32} & \phi_{33} \end{bmatrix}$$

Such an array is said to be semireducible. The examination of semireducible arrays will be left to another occasion.

Theorem V. If ϕ is a complete irreducible rectangular array such that

$$x_i\phi - \phi x_i = -X_i\phi + \phi Y_i$$

then both X_i and Y_i give matrix representations of the Lie algebra and of the Lie group.

The fact that ϕ is irreducible determines that X_i and Y_i are unique apart from scalar multiples of the identity matrix, which will not affect Poisson bracket expressions.

Evaluation of

$$[x_i, x_j]\,\phi - \phi[x_i, x_j]$$

shows that this is equal to

$$-[X_i, X_j]\,\phi + \phi(Y_i, Y_j]$$

The proof of the theorem follows from this.

Theorem V seems to suffice to yield all the matrix representations of the familiar continuous groups, although the sufficiency has not yet been demonstrated.

For the basic spin representation of the three-variable rotation group put

$$\phi = \begin{bmatrix} u, & v+iw \\ v-iw, & -u \end{bmatrix}$$

whence

$$\begin{aligned} u\phi - \phi u &= -U\phi + \phi U \\ v\phi - \phi v &= -V\phi + \phi V \\ w\phi - \phi w &= -W\phi + \phi W \end{aligned}$$

where

$$U = \tfrac{1}{2}\begin{bmatrix} i & \\ & -i \end{bmatrix} \qquad V = \tfrac{1}{2}\begin{bmatrix} & i \\ i & \end{bmatrix} \qquad W = \tfrac{1}{2}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$$

For the rotation group in $n = 2\nu + 1$ variables, consider the Clifford algebra of a frame of anticommuting matrices $X_1, X_2, \ldots, X_n$ with $X_i^2 = I$.

The Lie algebra has $\frac{1}{2}n(n-1)$ basal elements x_{ij}, satisfying (distinct letters denoting distinct suffixes)

$$x_{ij} = -x_{ji}, \qquad [x_{ij}, x_{kp}] = 0, \qquad [x_{ij}, x_{jk}] = x_{ik}, \qquad [x_{ij}, x_{ij}] = 0$$

Then clearly X_iX_j commutes with X_kX_p. Also

$$\begin{aligned} [X_iX_j, X_jX_k] &= X_iX_j^2X_k - X_jX_kX_iX_j = X_iX_k - X_kX_jX_j^2 \\ &= X_iX_k - X_kX_j = 2X_iX_k \end{aligned}$$

and

$$[X_iX_j, X_iX_j] = 0$$

Thus

$$x_{ij} \to \tfrac{1}{2}X_iX_j$$

gives a matrix representation of the Lie algebra. It is in fact the basic spin representation. It is, rather remarkably, an explicit form for this representation. The usual methods of obtaining the basic spin representation do not usually give it in explicit form, although Joos (1959) and Dodds (1973) have obtained explicit forms in terms of the elements of the orthogonal matrix.

This representation was obtained directly, without reference to Theorem V. The connection with the theorem, however, is easily obtained, and this connection is important for obtaining the higher degree spin representations.

Put

$$\phi = \sum x_{ij}X_iX_j$$

summed for all combinations of pairs of suffixes, i, j.

Then

$$\begin{aligned} x_{ij}\phi - \phi x_{ij} &= \sum [x_{ij}, x_{jk}]X_jX_k + \sum [x_{ij}, x_{ik}]X_iX_k \\ &= \sum x_{ik}X_jX_k - \sum x_{jk}X_iX_k \\ &= \tfrac{1}{2}(X_iX_j\phi - \phi X_iX_j) \end{aligned}$$

For the true representations of the orthogonal group in $n = 2\nu + 1$ variables put

$$\phi = [x_{st}]$$

Then

$$[x_{ij}, \phi] = [n_{ij}, \phi]$$

where

$$n_{ij} = [\delta_{is}\delta_{jt} - \delta_{it}\delta_{is}]$$

with $\delta_{ii} = 1$, $\delta_{ij} = 0$, $(i \neq j)$.

This gives the basic n-rowed orthogonal representation. The symmetrized invariant matrices of ϕ yield the other true representations.

For the full linear groups, the matrix representations are derived easily from Theorem V. The basic Lie algebra elements may be denoted by x_{ij}, where this corresponds to a matrix with unity in the ith row, jth column, with zeros elsewhere. Then if

$$\phi = [x_{st}]$$

clearly

$$x_{ij}\phi - \phi x_{ij} = e_{ij}\phi - \phi e_{ij}$$

where e_{ij} again denotes the matrix with unity in the ith row, jth column with zeros elsewhere, but this time representing a scalar matrix and not a Lie algebra element.

This yields the basic representation of the Lie algebra. The other representations are derived directly from the symmetrized invariant matrices of ϕ.

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